Climb of a bore on a beach.
Part 1. Uniform beach slope

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The shoreward travel of a bore into water at rest on a beach of uniform slope is studied to elucidate why, in a class of problems—mainly gas-dynamical ones involving non-uniform shock propagation—similarity solutions seem to act like magnets attracting other solutions. For the shallow-water problem, the real magnet is shown to be the shore singularity of the governing differential equations. The shore singularity of the solution is shown to be a directional singularity of the water acceleration, for a fairly wide range of conditions, and a rather detailed asymptotic approximation for the bore development near shore is deduced.

1. Introduction

Interest has been shown in recent years in a class of gas-dynamical problems involving the non-uniform propagation of shock waves, which present difficulties on account of both their strong non-linearity and their awkward boundary conditions. To date, virtually the only bona fide analytical solutions available for this class of problems are the 'similarity' solutions, developed extensively by Sedov and his school, and typified by Guderley's (1942) now famous solution for the converging cylindrical shock. This was obtained by looking, not for the solution satisfying boundary conditions of a type known to be necessary and sufficient for the determination of a solution, but by looking only for a particular solution depending, rather than on radius and time separately, on a combination of both such as $(t^{-n}r)$. That such a particular solution should be of use for boundary conditions differing appreciably from those it happens to satisfy is not immediately obvious, but numerical computation (Payne 1957) of some relevant cases yielded solutions which, despite marked differences in their initial behaviour, soon converge very closely to Guderley's. It therefore appears that, for some range of initial conditions, the gas motion forgets its initial conditions, and it is of academic and practical interest to understand a little how it comes about in such very non-linear problems.

Now, Keller, Levine & Whitham (1960) have computed three comparison cases of bores travelling shoreward into water at rest on a beach of uniform slope, and have shown how the solutions, while initially rather different, converge towards each other as the bore approaches the shore. Their 'shallow water'

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problem is therefore an analogue of the gas-dynamical problems showing forgetfulness, and, being much the simplest analogue known to us, is used in the following as the example in whose context forgetfulness will be studied.

Our approach is based on the observation that non-trivial similarity solutions have a singularity, as is obvious from the form of the independent variable, and, in the gas-dynamical similarity solutions, that singularity occurs usually at a *singularity of the governing differential equations*. In fact, those equations are mostly variants of the Euler–Poisson–Darboux equation,

$$\partial^2\phi/\partial y^2 - \partial^2\phi/\partial z^2 - kz^{-1}\partial\phi/\partial z = 0 \quad (k = \text{const.}), \tag{1}$$

which is called singular at $z = 0$. It will emerge below that the phenomenon of forgetfulness is due to this singularity, rather than any non-linearity, of the equations.

The field of singular partial differential equations is not yet well explored. Mathematical students of the subject may be attracted to the 'singular Cauchy problem', i.e. the question of what boundary conditions prescribed on the singular line $z = 0$ are necessary and sufficient for the existence, uniqueness and stability of solutions. Physically, however, this problem may be absurd. In shallow-water theory, for instance, $z = 0$ corresponds to zero water depth, and who would try to control the sea by doing something to the water at a place where there is, in fact, no water? The same remark applies in gasdynamics when the singularity corresponds to vacuum; the mathematical trap is not so obvious when the singularity corresponds to an axis of symmetry, but it is still clear that a direct physical realization of boundary conditions on such an axis is not possible. It is therefore of the essence of singular partial differential equations, where they arise from physics, that they occur with boundary conditions prescribed at a distance from the singularity of the equations. To bridge this gap is one of the basic mathematical tasks.

Perhaps, it should be stressed still further that the main difficulty turns out to arise from the boundary conditions, rather than from the non-linearity (which will be removed, temporarily, in §2) of the differential equations. Part of the boundary conditions are furnished by the bore relations (§2), which represent a *floating* boundary condition of awkward algebraic form. The rest is furnished by a seaward boundary condition (§§3, 4), which is firm and linear, in the formulation adopted below—but which is just that boundary condition which the solution is expected to forget. A second difficulty arises from the fact that the solutions of (1) can have uncommonly complicated singularities at $z = 0$.

The tool used below for bridging the gap between the seaward boundary condition and the singularity of the governing equations is the theory of structure of equations of the type of (1). It has been developed fairly completely (Meyer 1949, 1958) for the regular equation ($z \geqslant \epsilon > 0$), and is applied here with the aim of obtaining restrictions on the possible qualitative behaviour of all solutions in the regular region ($z > 0$) sufficient to leave only one singularity admissible.

To achieve this, a monotoneity assumption concerning the seaward boundary condition is introduced in §4. Physically, it amounts to a somewhat intricate statement regarding the signature of the fluid acceleration. We conjecture that it is a sufficient, rather than a necessary, assumption for the results we deduce, but also that it contains some of the essence of the assumption necessary to ensure the phenomenon of forgetfulness. In the laboratory, the bore might be generated by pushing a wave-maker piston into water at rest, and the experimenter would be at liberty to continue the piston motion at will and thereby influence the development of the bore. Indeed, by withdrawal of sufficient water, he might prevent the bore from reaching the shore. The need for an inequality concerning the acceleration at the seaward boundary is thus plausible. We also subject the seaward boundary conditions to certain regularity assumptions (§§3, 4), partly on straightforward physical grounds, and partly to keep the discussion within the common language of elementary calculus. Finally we add, in §3, the (mathematically illegitimate) assumption that the bore reaches the shore at a finite time; its relation to the monotoneity assumption will not be studied here, but we show in §7, by comparison with the numerical results of Keller *et al.*(1960), that our assumptions are compatible.

From these assumptions, we deduce in §5 certain qualitative properties of limits at the shore, which are shown in §6 to determine the solution near the shore also quantitatively to a notably high approximation. It exhibits forgetfulness in a striking manner, since the first approximation to the non-dimensional bore description that contains a parameter depending on the seaward boundary condition is the approximation of the seventh order! None the less, the concept of forgetfulness emerges as a somewhat misleading one. The strong role played by the monotoneity assumption in our proof indicates that the solution remembers, and is indeed critically determined by, certain *qualitative* properties of the seaward boundary condition. It is only the quantitative detail of that boundary condition which the (non-dimensional) solution all but forgets. Moreover, the basic velocity scale is found to depend on the seaward boundary condition; the basic acceleration scale, by contrast, depends only on the beach slope.

2. Governing equations

A straight bore is assumed to travel in the direction of x increasing into undisturbed water of depth $h_0(x)$. The water motion behind the bore is assumed governed by the 'first-order shallow-water' equations (Stoker 1957), according to which the vertical water velocity is neglected, the horizontal water velocity u in the direction of x increasing does not vary in the vertical direction, and the total water depth $h(x,t)$ (figure 1) is related to u by

$$\partial h/\partial t + \partial(hu)/\partial x = 0, \tag{2}$$

$$\partial u/\partial t + u\,\partial u/\partial x + g\,\partial(h-h_0)/\partial x = 0, \tag{3}$$

where g denotes the gravitational acceleration. These equations are not valid within the bore itself, but if the bore be regarded as a discontinuity of the motion such that u jumps from 0 to u_b, and h from h_0 to $h_b > h_0$, then these discontinuities are related to the bore velocity

$$dx_b/dt_b = V \tag{4}$$

by (Stoker 1957)

$$u_b/V = 1-h_0/h_b, \tag{5}$$

$$2V^2 = gh_b(1+h_b/h_0). \tag{6}$$

Our primary concern is with the bore motion in the vicinity of the initial shore line $h_0 = 0$, and it will be shown in Part 2 that this motion is, to a first approximation, independent of the beach shape, provided the beach slope is non-zero and finite. We need not hesitate, therefore, to avail ourselves in Part 1 of the drastic formal simplification afforded by the assumption of uniform beach slope. Let

$$h_0(x) = -\gamma x/g \quad (\gamma = \text{const.} > 0), \tag{7}$$

$$c^2 = gh(x,t) \quad (c \geqslant 0), \tag{8}$$

$$\alpha = 2c+u+\gamma t-u_0, \quad \beta = 2c-u-\gamma t+u_0, \tag{9}$$

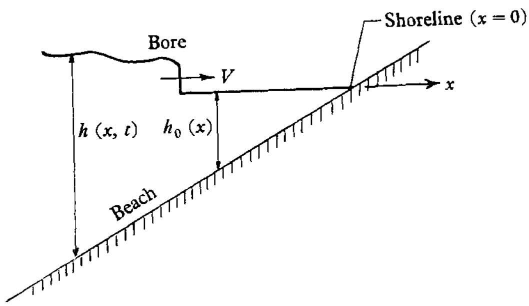

FIGURE 1. Definition sketch (a greatly contracted horizontal scale being implied).

where u_0 is a constant to be chosen presently. Then from (2) and (3) we have

$$\alpha = \text{const.} \quad \text{on the lines} \quad dx/dt = u+c, \tag{10}$$

$$\beta = \text{const.} \quad \text{on the lines} \quad dx/dt = u-c, \tag{11}$$

called respectively advancing and receding characteristic lines.

It is convenient to employ the characteristic parameters α, β as independent variables. The legitimacy of this transformation is not obvious *a priori*, but we may proceed formally, and it will emerge by and by that the transformation is regular at the positions and times under consideration. If partial differentiation with respect to α and β be denoted by subscripts, (10) and (11) become

$$x_\beta = (u+c)\,t_\beta, \quad x_\alpha = (u-c)\,t_\alpha, \tag{12}$$

and cross-differentiation yields (Carrier & Greenspan 1958)

$$t_{\alpha\beta}+\tfrac{3}{2}(\alpha+\beta)^{-1}(t_\alpha+t_\beta) = 0, \tag{13}$$

a particular case of (1). It is notable that (13) also arises for waves on shallow water of constant depth h_0, but of course, the interpretation is different in the present case, where

$$u+\gamma t-u_0 = \tfrac{1}{2}(\alpha-\beta), \quad c = \tfrac{1}{4}(\alpha+\beta) \tag{14}$$

complete the system (12). The canonical equations of (13) are

$$a_\beta = -\tfrac{3}{2}(\alpha+\beta)^{-1}b, \quad b_\alpha = -\tfrac{3}{2}(\alpha+\beta)^{-1}a, \tag{15}$$

where

$$a = (\alpha+\beta)^{\frac{3}{2}}t_\alpha, \quad b = (\alpha+\beta)^{\frac{3}{2}}t_\beta. \tag{16}$$

The bore relations (5), (6) furnish floating initial conditions for (15). Both the uniqueness theorem (Friedrichs 1954) and physical considerations indicate that a further boundary condition specifying information on the water motion *behind* the bore is required, and that at least some of its features must be anticipated to influence the bore motion appreciably. On the other hand, it will emerge in the next section that the amount of seaward information required is limited.

3. The limiting characteristic

The assumption that the bore reaches the shore in a finite time has some drastic implications, to be discussed now. Time will be measured from the moment at which the bore reaches the shore, and our concern is then only with $t \leqslant 0$, and†

$$c_b = (gh_b)^{\frac{1}{2}} \rightarrow 0 \quad \text{as} \quad t \rightarrow 0, \tag{17}$$

$$h_b > h_0 \quad \text{for} \quad t < 0. \tag{18}$$

This implies, by (5), (6) and (8),

$$u_b + c_b > V > c_b > 0, \quad V > u_b > 0 \quad \text{for} \quad t < 0. \tag{19}$$

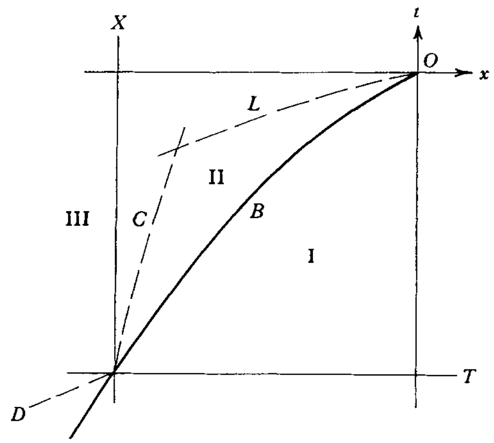

FIGURE 2. Diagram of (x, t)-plane showing locus of successive bore positions (bore initially supercritical).

The successive bore positions $x_b(t)$ may be traced in an (x, t)-plane (figure 2). On physical grounds, the bore height $h_b - h_0$ will be assumed a single-valued continuous function of t, for $t < 0$, except at times at which two bores merge; but if t_1 be the last negative time at which that occurs, we may limit our attention to $t_1 < t \leqslant 0$. By (6), $V(t)$ is then also single-valued and continuous, and so are $u_b(t)$ and $c_b(t)$, by (5) and (8). Given any $t < 0$, it therefore follows from (19) and (10) that the bore meets at that time just one advancing characteristic line of the water motion behind it, and the limit $t \uparrow 0$ defines a limiting member L of this family of characteristic lines. The importance of this 'limiting characteristic' was first realized by Guderley (1942), and the name is even more appropriate than has been generally appreciated. In figure 2, the region I corresponding to water at rest and the region between L and the bore path B contain all the points

† A suffix b will be used throughout to denote the limit of a quantity as the bore is approached from the seaward side.

(x, t) from which the bore path can be reached by following a characteristic line in the sense of increasing time. The water motion at points in other regions can have no influence on the bore development for $t < 0$, and is thus irrelevant to the present investigation. In what follows, the 'value' of any quantity at a 'point' on L can therefore, like the point itself, have a meaning only as a limit approached by some limiting process in which L is approached from the region between B and L.

For definiteness, assume the bore to be known up to some negative time $T > t_1$, and let C denote the receding characteristic line of the water motion behind the bore which issues from the bore at time T (figure 2, which is drawn for the 'supercritical' case $u_b > c_b$). Then (Friedrichs 1954) knowledge of u and h on, and only on, the segment of C between B and L (figure 2) is necessary for the unique determination of the bore development for $T < t \leqslant 0$, and this segment of C may be called the (mathematical) seaward boundary. The position of L in the (x, t)-plane is not, however, known in advance, so we must assume $\beta_b(T)$ to be known and α to be given as a function $\alpha_c(t)$ on C over a sufficient interval $T \leqslant t \leqslant T'$.

Observe that no generality is lost in taking the water motion to be bore-free in the interior of the region II (figure 2) bounded by C, B and L. To see this, begin by supposing an advancing† bore enters (in the sense of increasing time) this region from its boundary. That it enters across L is incompatible with the definition of the limiting characteristic, since $V \leqslant u+c$ on the shoreward side of an advancing bore. The same inequality excludes the bore's leaving the region across L. It enters therefore across C and either merges with the original bore, which is also advancing, or peters out at $t < 0$. A merger at $t = 0$ would leave the existence of L unimpaired, i.e. the second bore would not penetrate into the interior of the region II of the original bore. But a merger at $t < 0$ is excluded by the condition $T > t_1$, and a bore petering out at $t < 0$ can be similarly removed from consideration by a reduction of $|T|$.

Next, suppose a receding bore enters region II from the boundary with non-zero strength. It cannot do so from B, due to the absence of advancing bore mergers. But if it enters across C, then it must leave across L, or peter out at $t < 0$, and can thus also be removed from consideration by a reduction of $|T|$. Finally, the same arguments apply also to any bore forming in the interior of region II or, with zero strength, on its boundary. We therefore postulate $|T|$ to be chosen so that the water motion is bore-free in the interior of region II.

It follows that $\alpha_c(t)$ is a continuous function and, if our assumptions are consistent, (2) and (3) must possess a continuous solution in the interior of region II. It is, in fact, uniformly continuous, since the continuity of α on C implies the boundedness of α 'on L', by (10), and it follows from (9) and (17) that u_b must tend to a finite limit as $t \to 0$ (Keller *et al.* 1960), which will be identified with the constant u_0 of (9), so that

$$\alpha = 0, \quad \text{and} \quad \beta \to 0 \quad \text{as} \quad t \to 0, \quad \text{'on L'}, \tag{20}$$

$$u_b \to u_0, \quad \alpha_b \to 0, \quad \beta_b \to 0 \quad \text{as} \quad t \to 0 \quad \text{on the bore.} \tag{21}$$

† That is a bore across which the water height rises from the shoreward to the seaward side.

It follows from (5) and (6) that h_b cannot remain positive when $h_0 = 0$, i.e. h_b and h_0 must vanish together (Keller *et al.* 1960), as indicated in figure 2. Whether the shore line $h(x,t) = 0$ moves beyond its initial position at $t > 0$, will be studied in Part 3.

Moreover, if our assumptions are physically consistent, the solution of (2) and (3) must be single-valued in the interior of region II, since (Mahony 1956) multi-valuedness could be removed only by the appearance of a second bore. A general existence proof is beyond the scope of the present study, but (§7) the numerical results of Keller *et al.* (1960) confirm the consistency of our assumptions in some cases, at least.

4. A monotoneity assumption

To facilitate the calculation of values on L, we assume $dt/d\alpha_c$ to be a piecewise continuous function of α_c. The same follows for a from (16), and by (15) and (16):

$$\alpha_\beta / t_\beta = -3c^{\frac{1}{2}}, \tag{22}$$

so that a remains a piecewise continuous function in region II for $t \leqslant 0$. The bore conditions (5) and (6) imply a relation between a and b on the bore, which can be used (Ho & Meyer 1962) to show that the piecewise continuity of a_b implies that

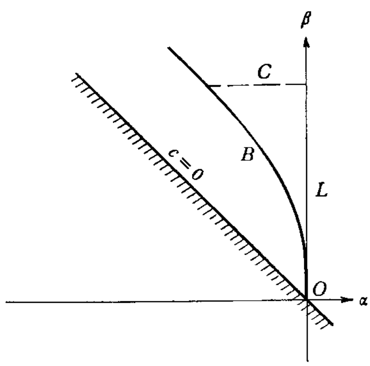

FIGURE 3. Diagram of characteristic plane.

of b_b for $t < 0$, and by (15) and (16), b remains also piecewise continuous in region II. With the choice of T at our disposal, $a(\alpha,\beta)$ and $b(\alpha,\beta)$ may therefore be considered uniformly continuous in region II, for $t < 0$, and a even for $t \leqslant 0$. To simplify a proof in §5, we shall also assume there that $d^2t/d\alpha_c^2$ be piecewise continuous, even though a somewhat weaker assumption would suffice (Ho & Meyer 1962).

Our main assumption, however, is that $\alpha_c(t)$ is strictly increasing. It implies, first of all, that $t_\alpha > 0$ everywhere in the interior of region II. Indeed, suppose a point P occurs where $t_\alpha < 0$. Then since $t_\alpha \geqslant 0$ on C and by (22), P must be preceded on the same advancing characteristic line by a point where $t_\alpha = 0$. And similarily, a point where $t_\alpha = 0$ must be followed by one where $t_\alpha < 0$. This would imply (Meyer 1949) the occurrence of a 'limit line' $t_\alpha = 0$ and the multi-valuedness of the solution, in contradiction to the conclusions of the preceding section.

This argument applies also to the seaward boundary itself and shows $t_\alpha \neq 0$ on C, so that the monotoneity assumption is seen to imply the differentiability of $\alpha_c(t)$ and

$$d\alpha_c/dt > 0. \tag{23}$$

This implies, in turn, that

$$d\alpha_b/dt > 0 \quad \text{for} \quad t < 0, \tag{24}$$

since the contrary would require the intersection of two advancing characteristic lines carrying different values of α, and hence, multi-valuedness of the solution. Now, as $h_b \downarrow 0$, also $(\alpha_b+\beta_b)\downarrow 0$, by (8) and (14), but $\alpha_b \uparrow 0$, by (24) and (21), so the bore path in the characteristic plane (figure 3) must approach the origin from a direction such that the bore slope $d\alpha_b/d\beta_b = \alpha_b'$ satisfies

$$-1 \leqslant \lim_{c\to 0} \alpha_b' \leqslant 0. \tag{25}$$

5. Limits at the shore

The last inequality may be used to confirm the assumption

$$\lim_{c\to 0} u_b = u_0 > 0 \tag{26}$$

of Keller *et al.* (1960). Indeed, by (14), (4), (8) and (7),

$$\frac{du_b}{dc_b} = -2\frac{1-\alpha_b'}{1+\alpha_b'} - \gamma\frac{dt_b}{dc_b} = -2\left[\frac{1-\alpha_b'}{1+\alpha_b'} - \frac{c_b}{V}\frac{dh_0}{dh_b}\right].$$

At sufficiently short times before the bore reaches the shore, $h_b > h_0$ and hence both $c_b/V < 1$, by (6) and (8), and $dh_0/dh_b \leqslant 1$, since h_b and h_0 vanish together (§3). By (25), therefore, $du_b/dc_b < 0$ at such times, i.e. u_b increases ultimately, and (26) now follows from (19).

A number of other limits† follow (Keller *et al.*) from (26). By (18),

$$0 \leqslant \lim\,(h_0/h_b) < 1,$$

and so from (5), (6) and (8),

$$\lim\,(gh_b^2/h_0) = \lim\,[c_b^4/(gh_0)] = 2u_0^2, \tag{27}$$

$$\lim V = u_0, \quad \lim\,[(V-u_b)/c_b^2] = (2u_0)^{-1}. \tag{28}$$

By (4) and (7), (27) and (28) imply also

$$\lim\,(c_b^{-4}x_b) = -\lim\,(c_b^{-4}gh_0/\gamma) = \lim\,(u_0c_b^{-4}t_b) = -(2\gamma u_0^2)^{-1}. \tag{29}$$

The main contribution of the present investigation is the

Lemma: $\lim a(0,\beta) \neq 0.$

The existence of this limit follows from the uniform continuity of a (§4). Since $t_\alpha > 0$ in the interior of region II (figure 2), t_α and a must be non-negative on L, and since a zero of a on L would, by (22), be followed promptly by negative values,

$$a(0,\beta) > 0 \quad \text{for} \quad t < 0, \tag{30}$$

and the Lemma implies

$$\lim a(0,\beta) = a_0 > 0, \tag{31}$$

† Unless the contrary is indicated, limits will now be understood taken as $c \to 0$.

where from the structure of the shore singularity will be deduced at the end of this section.

To establish the Lemma, we begin by strengthening (25). On the bore,

$$t_\beta - \alpha_b' t_\alpha = c_b^{-1}(V-u_b)\, dt_b/d\beta,$$

by (12) and (4); on the other hand, $dt_b/d\beta = \alpha_b' t_\alpha + t_\beta$, and so by (28),

$$\lim\, [t_\beta d\beta/dt] = \lim\, [t_\alpha d\alpha/dt] = \tfrac{1}{2} \tag{32}$$

on the bore. It follows, by (14) and (29), that

$$\lim\, (c^{-3} t_\alpha) = -(4\gamma u_0^3)^{-1} \lim\, (1+\alpha_b'^{-1}) \tag{33}$$

on the bore, if we suppose $\lim \alpha_b' \neq 0$. But this leads to a contradiction, as follows. By (13), (14) and (32), as $c \to 0$ on the bore,

$$\partial(c^{-2}t_\alpha)/\partial\beta \sim -\tfrac{1}{2}c^{-3}t_\alpha\{1+3(1+\alpha_b')/8\},$$
$$\partial(c^{-3}t_\alpha)/\partial\beta \sim -\tfrac{3}{4}c^{-4}t_\alpha\{1+(1+\alpha_b')/2\}.$$

Thus if $-1 < \lim \alpha_b' < 0$, then $c^{-3}t_\alpha$ tends to a finite positive limit, and $c^{-2}t_\alpha < 0$ on L (figure 3), for sufficiently small c. Again, if $\lim \alpha_b' = -1$, then $\lim\, c^{-3}t_\alpha = 0$, and $c^{-3}t_\alpha < 0$ on L, because (33) then shows $c^{-4}t_\alpha \sim (\gamma u_0^3)^{-1} c^{-1} dc/d\beta \to +\infty$ on the bore, as both c and $\beta \downarrow 0$. Both conclusions contradict (30), and so (25) must be replaced by

$$\lim \alpha_b' = 0, \tag{34}$$

and the image of region II (figure 2) in the characteristic plane must indeed be as shown in figure 3.

It follows, by (16), (32) and (29), that

$$\lim\, (c_b^{-9/2} b_b) = -2(\gamma u_0^3)^{-1}, \tag{35}$$

$$\alpha_b' a_b = O(b_b) = O(c^{9/2}). \tag{36}$$

Note also that, since $t(0,\beta) \to 0$ on L,

$$\lim \beta^{-\frac{1}{2}}\, b(0,\beta) = 0, \tag{37}$$

by (16). For fixed $\beta > 0$, moreover, (15) shows $b_\alpha(\alpha_1,\beta)$ to exist for $\alpha_b < \alpha_1 < 0$, so that

$$\begin{aligned} b(\alpha_b,\beta) - b(0,\beta) = \alpha_b b_\alpha(\alpha_1,\beta) &= -\tfrac{3}{2} a(\alpha_1,\beta)\,\alpha_b(\alpha_1+\beta)^{-1} \\ &= O[a(\alpha_1,\beta)\,\alpha_b'], \end{aligned} \tag{38}$$

and b is, like a, uniformly continuous in region II for $t \leqslant 0$.

Now consider the integral

$$I = \int_0^\xi \int_\eta^{\eta_1} a_{\alpha\beta}\, d\beta\, d\alpha = a(0,\eta) - a(\xi,\eta) + \int_0^\xi a_\alpha(\alpha,\eta_1)\, d\alpha$$

over the characteristic rectangle PQQ_1P_1 of figure 4. By (34), the point P on B can be chosen so that $|\xi/\eta| \ll 1$ for $\eta/u_0 \ll 1$, and then by (15), (16),

$$I = \frac{3}{2}\int_0^\xi \int_\eta^{\eta_1} (\tfrac{3}{2}a+b)(\alpha+\beta)^{-2}\, d\beta\, d\alpha = O(\alpha_b'),$$

since
$$\int_0^\xi \int_\eta^{\eta_1} (\alpha+\beta)^{-2}\, d\beta\, d\alpha = \frac{\xi}{\eta}\left(1-\frac{\eta}{\eta_1}\right) + O(\xi^2/\eta^2) = O(\alpha_b').$$

By (18),

$$a_\alpha = \tfrac{3}{2}(\alpha+\beta)^{-1}a + (\alpha+\beta)^{\frac{3}{2}}\partial^2 t/\partial\alpha^2,$$

and η_1 may be chosen so that P_1Q_1 is a segment of the seaward boundary C, where a_α is seen to be bounded, if we now appeal to the piecewise continuity of $d^2t/d\alpha_c^2$. Hence,

$$a(\alpha_b(\eta),\eta) - a(0,\eta) = O(\alpha_b'). \tag{39}$$

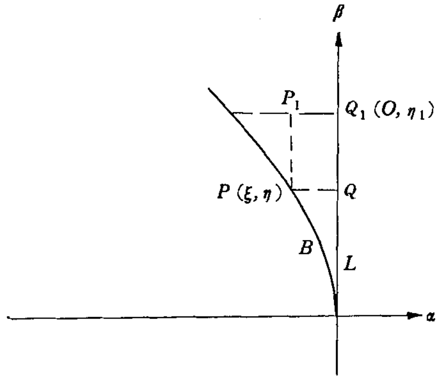

FIGURE 4. Definition of rectangle PQQ_1P_1 in the characteristic plane.

Finally, let $\beta^{-\frac{1}{2}}b(0,\beta) = f(\beta)$. Then by (37) and (15),

$$\lim \frac{a(0,\beta)-a_0}{b(0,\beta)} = \lim\,(a_\beta/b_\beta) = -\tfrac{3}{2}\lim\frac{b}{\beta b_\beta}$$
$$= -3\lim\frac{f}{f+2\beta f'},$$

if it exists. Now suppose $\lim a(0,\beta) = a_0 = 0$. Then (30) and (15) imply $f(\beta) < 0$ for sufficiently small β, and the same follows for $f'(\beta)$ from (37). Hence,

$$\lim a(0,\beta)/b(0,\beta)$$

exists and $a\,(0,\beta) = O[b(0,\beta)]$, at most. But by (36) and (38) $a_b'^{-1}b(0,\beta)\to 0$, since $\alpha_1 \to 0$ together with α_b and we are supposing $a \to 0$. Therefore (39) reduces to $a_b = O(\alpha_b')$, and from (36), $a_b = O(c^{9/4})$. It follows from (15) and (36) that

$$\partial(c^{-2}a)/\partial\beta = -c^{-3}a_b\{\tfrac{1}{2} + O(\alpha_b')\} \to -\infty$$

on the bore, while $c^{-2}a_b \to 0$, and that is incompatible with the conclusion that $t_\alpha > 0$ everywhere in region II (§4). Hence, $a_0 \neq 0$.

As a corollary,

$$\lim\,(c^{-9/2}\alpha_b') = -2(\gamma a_0 u_0^3)^{-1} \tag{40}$$

now follows from (32) and (35), so that the bore approaches the limiting characteristic line L very closely in the characteristic plane (figure 3). Moreover, from (35), (38) and (40),

$$b(0,\beta) = O(c^{9/2}),$$

and so, by (15), also

$$a(0,\beta) - a_0 = O(c^{9/2}),$$

and from (39),

$$a_b - a_0 = O(c^{9/2}). \tag{41}$$

The physical interpretation of the Lemma is that the shore singularity of the water motion behind the bore—in the region II here studied—is a singularity of the acceleration and has a markedly directional character. Indeed, t_α^{-1} and t_β^{-1} have the dimension of the fluid acceleration and may be loosely described as 'characteristic acceleration components', from which $\partial u/\partial t$ and $\partial u/\partial x$ may be obtained as

$$\partial u/\partial t = (4c)^{-1}[(u+c)\,t_\alpha^{-1} + (u-c)\,t_\beta^{-1}] - \gamma,$$
$$\partial u/\partial x = (4c)^{-1}(t_\alpha^{-1} + t_\beta^{-1}).$$

By (16), (29), (35) and the Lemma

$$t_\alpha^{-1} = O(a_0^{-1} c^{\frac{3}{2}}), \quad t_\beta^{-1} = O(c^{-3}) = O(t^{-\frac{3}{4}}),$$

i.e. the purpose of the Lemma has been to deduce the absence of a shore singularity of t_α^{-1} from the monotoneity assumption that $t_\alpha^{-1} > 0$ on the seaward boundary.

6. Approximate bore path

The results of the preceding section determine a quite detailed quantitative approximation for the solution near the shore. To trace the bore path, it is convenient to employ the parameter

$$z = c_b/(u_b + 2c_b) \tag{42}$$

related to the 'bore strength' $M-1 = (V-c_b)/c_b$ and surface elevation ratio $H = h_0/h_b$ by

$$z^{-1}(1-2z) = u_b/c_b = M(1-H), \tag{43}$$
$$M^2 = \tfrac{1}{2}(1+H^{-1}), \tag{44}$$

according to (5), (6) and (8). At the shore, $z = 0$, by (17) and (26), and it is seen from (43), (44) that z increases monotonically from 0 to only $\frac{1}{2}$, as M^{-1} increases from 0 to 1†. A complete description of the bore development will be obtained if x_b and t_b are expressed in terms of z, and since

$$x_b = -\gamma^{-1} H c_b^2, \tag{45}$$

by (7) and (8), and

$$\gamma t_b = -\int_0^z V^{-1} \frac{d}{dz}(Hc_b^2)\,dz = -\frac{H}{M}c_b - \int_0^z M^{-2} H \frac{dV}{dz}\,dz$$
$$= -\frac{H}{M}c_b + \tfrac{1}{5}u_0 z^5 + O(z^6), \tag{46}$$

by (4), (43), (44) and (28), we proceed to calculate $c_b(z)$ from the first of (9), which may be written

$$c_b = z(u_0 + \alpha_b - \gamma t_b) = u_0 z + O(c_b^5), \tag{47}$$

by (42), (29) and (40). From (4), (12) and (16),

$$d\alpha_b/dt_b = (u_b + c_b - V)/(2c_b t_\alpha) = 4a_b^{-1} c_b^{\frac{1}{2}}(u_b + c_b - V),$$

and since (43), (44), (46) and (47) give $dt_b/dz = -2\gamma^{-1}u_0 z^3 + O(z^4)$, (28) and (41) lead to

$$\alpha_b = -\tfrac{16}{11}(\gamma a_0)^{-1} u_0^{\frac{5}{2}} z^{\frac{11}{2}} + O(z^{\frac{13}{2}}),$$

and

$$c_b/u_0 = z(1 - Hz/M)^{-1}[1 - \tfrac{1}{5}z^5 - \tfrac{16}{11}(\gamma a_0)^{-1} u_0^{\frac{3}{2}} z^{\frac{11}{2}} + O(z^6)]. \tag{48}$$

† But the usefulness of z stems even more from the opportunity it affords for eliminating certain disappointing series by the help of the explicit relations (43) and (44).

(A less practical, but in some ways more illuminating, representation is

$$\frac{\gamma x_b}{u_0^2} = -\frac{1}{2}\left(\frac{c_b}{u_0}\right)^4 \left[1 + 4\frac{c_b}{u_0} + \frac{23}{2}\left(\frac{c_b}{u_0}\right)^2 + 28\left(\frac{c_b}{u_0}\right)^3 + 59\left(\frac{c_b}{u_0}\right)^4 + \frac{502}{5}\left(\frac{c_b}{u_0}\right)^5 + \frac{32}{11}\frac{u_0^{\frac{3}{2}}}{\gamma a_0}\left(\frac{c_b}{u_0}\right)^{\frac{11}{2}} + O\left\{\left(\frac{c_b}{u_0}\right)^6\right\}\right],$$

obtained from (48) and (42) to (44) by straightforward expansion.)

The values of u_0 and a_0 depend on the seaward boundary condition; but that of a_0 is seen to have only a very small influence on the values of c_b, x_b and t_b near the shore. The curve of bore speed V *vs* shore distance x_b obtained from (48) is plotted in figure 5 for several values of a_0 with the left end of the curves corresponding roughly to a bore strength $M-1 = 0{\cdot}15$. Whitham's (1958) approximation (Keller *et al.* 1960) coincides with the curve for $a_0 = 0{\cdot}2\, u_0^{\frac{3}{2}}\gamma^{-1}$.

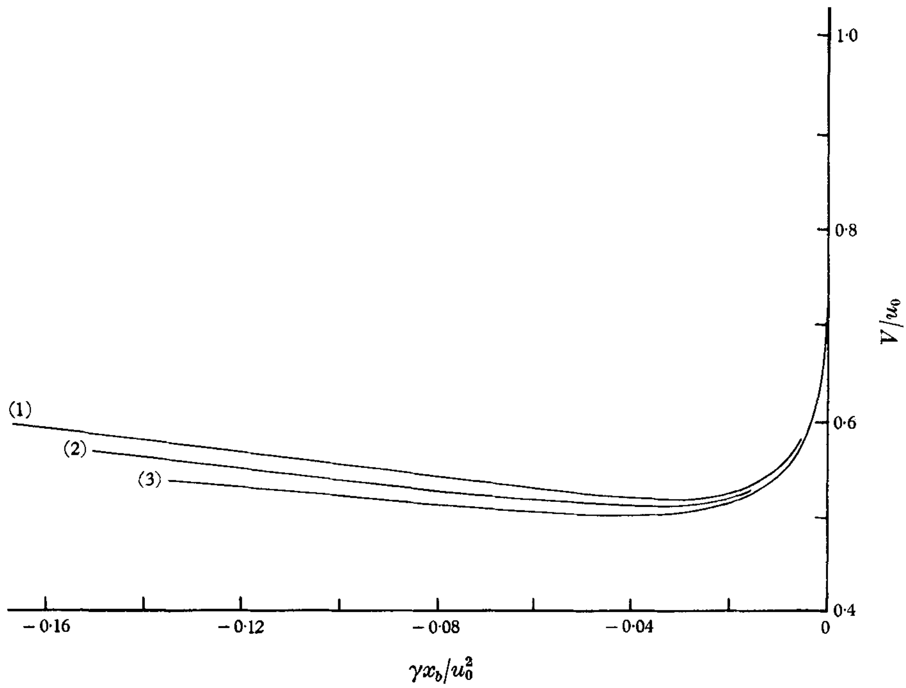

FIGURE 5. Variation of bore speed V with distance $-x_b$ from shore: $\frac{1}{2}\nu a_0/u_0^{\frac{3}{2}} = 1{\cdot}0$, $0{\cdot}1$ and $0{\cdot}05$, respectively, for curves (1), (2) and (3).

7. A particular seaward boundary

To connect the present investigation more closely with the computations of Keller *et al.* (1960), consider briefly the case studied by them in which the beach is flat for $x < X = x_b(T)$ and the motion of the bore is uniform for $t \leqslant T$.

If the bore is initially supercritical ($u_b > c_b > 0$), it is convenient to divide the region III of figure 2 into two regions III′ and III″ where, respectively, $x > X$ and $x < X$. The uniform motion in region III″ furnishes Cauchy data,

$$u = \text{const.} = u_1 \text{ and } c = \text{const.} = c_1 \text{ on } x = X,$$

for the motion in region III′. Since these, as well as (2) and (3), can be satisfied by $u = u(x)$, $c = c(x)$, the uniqueness theorem (Friedrichs 1954) shows the motion to be steady in region III′. From (9) and and (10), therefore,

$$\partial\alpha/\partial t = -(u+c)\,\partial\alpha/\partial x = \gamma, \tag{49}$$

whence

$$t_\alpha^{-1} = \partial\alpha/\partial t + (u-c)\,\partial\alpha/\partial x = 2\gamma c/(u+c), \tag{50}$$

in region III′. Since t_α is continuous across the boundary C of region II, the seaward boundary condition for region II is given by (50). By (2), hu = const. in region III′, so $u > 0$ and the monotoneity assumption is satisfied.

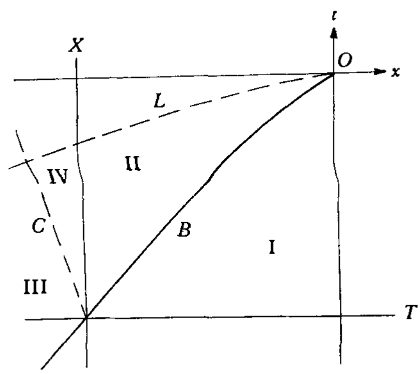

FIGURE 6. Diagram of (x, t)-plane in case where bore is initially subcritical (cf. figure 2).

If the bore is initially subcritical ($c_b > u_b > 0$), a receding 'simple wave' begins to propagate back into the region $x < X$, as soon as the bore crosses $x = X$, because (Stoker 1957) the beach is flat for $x < X$ and the water motion is uniform in the region III (figure 6) seaward of the receding characteristic line through (X, T). In view of the discontinuity of beach slope, the line $x = X$ is the proper seaward boundary for (2) and (3). In the simple wave region IV (figure 6) $u + 2c$ = const. (Stoker 1957), whence (49) follows again from (9) and (10). Since $x = X$ is not characteristic, $\partial\alpha/\partial x$ is continuous across it and the seaward boundary condition is again given by (50). Since $u_b > 0$ at $t = T$, the monotoneity assumption is satisfied at that time, and remains so until a second bore either crosses $x = X$ or forms at a limit point $t_\alpha = 0$ on $x = X$; neither occurs (Keller *et al.* 1960).

In both cases the results of Keller *et al.* confirm that the bore reaches the shore line within a finite time.

8. Whitham's rule

The results of §§5 and 7 also throw light on the background of a simple approximation rule for the climb of an advancing bore on a beach proposed by Whitham (1958). It is to apply the relation

$$du + 2dc - g(u+c)^{-1}\,dh_0 = 0, \tag{51}$$

valid on the advancing characteristic lines, to the values of the variables on the seaward side of the bore. An ordinary differential equation for M (or V) as function of h_0 is then obtained.

The original rule (51) cannot be tested directly, since it is not non-dimensional but the differential equation can be obtained as follows. By (43), (14) and (7), with $H = h_0/h_b$ again,

$$4M(2M^2-1)^{-1}h_0 dM/dh_0 = -1-(2\gamma M)^{-1}H(1+\alpha_b')/(\alpha_b' t_\alpha + t_\beta) \tag{52}$$

and by (4), (12) and (16),

$$\alpha_b' = d\alpha_b/d\beta = (u_b+c_b-V)(2c_b t_\alpha)^{-1}(\alpha_b' t_\alpha + t_\beta).$$

A second relation between α_b' and $dt_b/d\beta = \alpha_b' t_\alpha + t_\beta$ is obtained from differentiation of (43) and of the first of (14) along the bore, and α_b' and t_β may then be eliminated from (52) to obtain, after some manipulation, the exact differential equation for $M(h_0)$ in the form

$$\frac{4}{M-1}\frac{h_0 dM}{dh_0} + \frac{2(2M^2-1)(M^4+3M^3+M^2-\frac{3}{2}M-1)}{(M+1)(2M-1)^2(M^3+M^2-M-\frac{1}{2})} = \frac{-(2M+1)}{M(M+1)^2(2M-1)^2}\Gamma_b, \tag{53}$$

where

$$\Gamma = \gamma^{-1}\{t_\alpha^{-1} - 2\gamma c/(u+c)\}. \tag{54}$$

Note that $\gamma = -g dh_0/dx$ is the acceleration scale defined by the beach, and $2\gamma c/(u+c)$ is the local value of t_α^{-1} in water at rest or in steady flow (§7); Γ is therefore a wave strength ratio measuring the local strength of the advancing acceleration wave in the water.

Whitham's rule is equivalent (Keller *et al.* 1960) to neglect of the right-hand side of (53). Near the shore, by the Lemma of §5, $t_\alpha^{-1} = O(c^{\frac{3}{2}})$, and so $\Gamma_b = O(c_b)$, and since $M = O(c_b^{-1})$, the right-hand side of (53) is $O(c^5)$, while the second term on the left-hand side is $O(1)$. Whitham's rule must therefore furnish a good approximation, as soon as the bore comes at all close to the shore, in all cases to which the assumptions of §§3, 4 apply.

For the particular cases of §7, moreover, the seaward boundary condition (50) is $\Gamma = 0$, and (53) shows that Whitham's rule is then also a good approximation at the start of the bore's climb.

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